Fall 2014 Math 110 Review

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1 Limits

1) Find the following limit:

$$\lim_{\theta \to 0} \frac{1 - cos\theta}{sin\theta}$$

Notice that plugging in $\theta = 0$ directly gives an indeterminate form, so we must factor. Notice that

$$\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$$

So we can write

$$\lim_{\theta \to 0} \frac{1 - cos\theta}{sin\theta} = \lim_{\theta \to 0} \frac{1 - cos\theta}{\theta} \cdot \frac{\theta}{\sin\theta}$$

So limit becomes

$$\left(\lim_{\theta \to 0} \frac{1 - cos\theta}{\theta}\right) \left(\lim_{\theta \to 0} \frac{\theta}{\sin\theta}\right)$$

The limit on the left can be factored:

$$\lim_{\theta \to 0} \frac{1 - cos\theta}{\theta} = \lim_{\theta \to 0} \frac{1 - cos\theta}{\theta} \cdot \left(\frac{1 + cos\theta}{1 + cos\theta}\right)$$

$$= \lim_{\theta \to 0} \frac{1 - cos^2\theta}{\theta(1 + cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{sin^2\theta}{\theta(1 + cos\theta)}$$

$$= \left(\lim_{\theta \to 0} \frac{sin\theta}{\theta}\right) \cdot \left(\lim_{\theta \to 0} \frac{sin\theta}{(1 + cos\theta)}\right)$$

So the whole expression is

$$\left(\lim_{\theta \to 0} \frac{sin\theta}{\theta}\right) \left(\lim_{\theta \to 0} \frac{sin\theta}{(1 + cos\theta)}\right) \left(\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)}\right)$$

Which can now (finally) be evaluated:

$$= (1)\left(\frac{0}{1+1}\right)(1)$$

$$= 0$$

2) Find the following limit:

$$\lim_{x\to\infty} \frac{3x^3+6x^2}{2x^3-5x}$$

The procedure is to factor out the highest power of x from the top and bottom. In this case the highest powers are the same: 3.

$$= \lim_{x\to\infty} \frac{(x^3)(3+\frac{6}{x})}{(x^3)(2-\frac{5}{x^2})}$$

$$= \lim_{x\to\infty} \frac{(3+\frac{6}{x})}{(2-\frac{5}{x^2})}$$

The reason we do this is because $\lim_{x\to\infty} \frac{A}{x^k} = 0$ for any A, k. So we get:

$$= \frac{(3+0)}{(2-0)}$$

$$= \frac{3}{2}$$

2 Derivatives

1) Calculate g'(t)

$$g(t) = \sqrt{\frac{3t^2-1}{2-t^3}}$$

Rewrite g(t) using powers:

$$g(t) = \left(\frac{3t^2-1}{2-t^3}\right)^{\frac{1}{2}}$$

$$g'(t) = \left(\frac{1}{2}\right)\left(\frac{3t^2-1}{2-t^3}\right)^{\frac{-1}{2}} \cdot \frac{d}{dt}\left(\frac{3t^2-1}{2-t^3}\right)$$

$$g'(t) = \left(\frac{1}{2}\right)\left(\frac{3t^2-1}{2-t^3}\right)^{\frac{-1}{2}} \cdot \left(\frac{6t(2-t^3)-(-3t^2)(3t^2-1)}{(2-t^3)^2}\right)$$

2) Find $\frac{dy}{dx}$:

$$x^3+y^3=3xy$$

The idea here is to treat x and y each as functions, but whenever we differentiate a "y term," we multiply by $\frac{dy}{dx}$

Start by taking $\frac{d}{dx}$ of both sides:

$$\left(\frac{d}{dx}\right)(x^3 + y^3) = \left(\frac{d}{dx}\right)(3xy)$$

$$3x^2 + 3y^2 \cdot \left(\frac{dy}{dx}\right) = 3\left(1 \cdot y + x \cdot 1 \cdot \frac{dy}{dx}\right)$$

$$3x^2 + 3y^2 \cdot \left(\frac{dy}{dx}\right) = 3y + 3x\left(\frac{dy}{dx}\right)$$

Solve for $\left(\frac{dy}{dx}\right)$:

$$\left(\frac{dy}{dx}\right)(3y^2 - 3x) = 3y - 3x^2$$

$$\left(\frac{dy}{dx}\right) = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$\left(\frac{dy}{dx}\right) = \frac{y - x^2}{y^2 - x}$$

3 Integration

1) Solve the indefinite integral:

$$\int \frac{sec^2(\sqrt{x})}{\sqrt{x}} dx$$

The non-obvious substitution here is u = $\sqrt{x}$.

$$u = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = (\frac{1}{2})x^{\frac{-1}{2}}$$

$$2du = \frac{dx}{\sqrt{x}}$$

So the integral becomes

$$2\int sec^2 u \cdot du$$

and the derivative of $tanx$ is sec^2x, so

$$2\int sec^2 u \cdot du = 2tanx + C$$

And the last step is to back substitute

$$= 2tan(\sqrt{x}) + C$$

2) Solve the definite integral:

$$\int_0^1 x^9(x^5+1)^4 dx$$

Generally, we want to substitute the complicated for the simple. $(x^5+1)^4$ is complicated, so let's make it u^4 by:

$$u = x^5 + 1$$
$$\frac{du}{dx} = 5x^4$$
$$\frac{du}{5} = x^4 dx$$

This substitution is useful since, if we rearrange the integral slightly we get:

$$\int_0^1 x^5(x^5+1)^4 x^4 dx$$

So we can substitute *almost* everything:

$$\frac{1}{5}\int x^5 u^4 du$$

How can we get rid of the last x^5? Well, our substitution was $u = x^5 + 1$, so $x^5 = u - 1$!

$$\frac{1}{5}\int (u-1)u^4 du$$
$$= \frac{1}{5}\int (u^5 - u^4) du$$
$$= \frac{1}{5}\left(\frac{u^6}{6} - \frac{u^5}{5}\right)$$
$$= \frac{1}{5}\left(\frac{(x^5+1)^6}{6} - \frac{(x^5+1)^5}{5}\right)\Bigg|_0^1$$
$$= \frac{1}{5}\left[\left(\frac{2^6}{6} - \frac{2^5}{5}\right) - \left(\frac{1^6}{6} - \frac{1^5}{5}\right)\right]$$

4 Related Rates

A plane flying overhead is being tracked by a camera on the ground. If the plane is flying at a constant height of 2km and its horizontal distance to the camera is 4km, how fast is the camera turning (in radians/hour) if the plane is flying at a speed of 600km/h toward the camera?

Refer to figure 1 for the picture. As the plane flies, both θ and x are functions of time. They are related by the equation

$$tan\theta = \frac{x}{2}$$

We are looking for $\frac{d\theta}{dt}$, so we differentiate both sides **with respect to t**:

$$\frac{d}{dt}(tan\theta) = \frac{d}{dt}\left(\frac{x}{2}\right)$$

$$sec^2\theta\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{cos^2\theta}{2}\frac{dx}{dt}$$

The plane is flying at 600km/h. That is, $\frac{dx}{dt} = 600$. We also know that at this instant in time the plane is 4km away, horizontally, so $x = 4$. By trigonometry (refer to figure 1) $tan\theta = \frac{4}{2}$ so $cos\theta = \frac{2}{\sqrt{20}}$.

$$\frac{d\theta}{dt} = \frac{\left(\frac{2}{\sqrt{20}}\right)^2}{2} \cdot 600$$

$$\frac{d\theta}{dt} = 60\frac{rad}{hour}$$

5 Optimization

A landscape architect is asked to design a rectangular garden that is 150 square meters in area and is enclosed by a fence on all four sides. One side of the garden faces a street, and so the fence on this side is to be more ornate than the other sides; the cost of fencing for this side is $6 per meter. Each of the other three sides of the fence costs $3 per meter. What are the dimensions of the garden that will minimize the cost of the fencing?

Let x be the dimension which has one side costing $6 per meter, and one costing $3 per meter. Let y be the other dimension with two cheap sides (See figure 2).

Thus, the cost function is

$$C = 3x + 6x + 3y + 3y$$

$$C = 9x + 6y$$

We also have the requirement that the area, $A = xy$, is 150 squared meters. Thus

$$xy = 150$$

$$y = \frac{150}{x}$$

Combining these two equations, we get

$$C = 9x + 6\frac{150}{x}$$

$$C = 9x + \frac{900}{x}$$

We wish to *minimize* the cost, so we'll need to take a derivative!

$$C' = 9 - 900x^{-2}$$

The critical value(s) happen when C'=0:

$$0 = 9 - 900x^{-2}$$

$$x^2 = \frac{900}{9}$$

$$x = 10$$

where we have excluded the solution x = -10 since we can't physically have a negative length (even if it is the most cost efficient!) We can check that this is indeed a minimum by the second derivative test (it is a minimum if $C''(10) > 0$):

$$C'' = 1800x^{-3}$$

$$C''(10) = 1.8 > 0$$

So, the ideal x coordinate is 10 meters. This means that y should be

$$y = \frac{150}{x} = \frac{150}{10} = 15$$

And so

$$x = 10, y = 15$$

are our optimal coordinates.

6 Definition of the Derivative

Compute the derivative of $f(x) = \sqrt{3x+1}$ using the definition of the derivative.

The general definition of the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So for $f(x) = \sqrt{3x+1}$ it becomes

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h}$$

Multiplying by the conjugate,

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}}$$

We end up with

$$f'(x) = \lim_{h \to 0} \frac{3x + 3h + 1 - 3x - 1}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})}$$

Simplifying, we see that almost everything in the numerator cancels, *including the h* in 3h.

$$f'(x) = \lim_{h \to 0} \frac{3}{(\sqrt{3(x+h)+1} + \sqrt{3x+1})}$$

This cancellation of h is crucial, because it takes our limit out of indeterminate form and allows us to plug h in directly:

$$f'(x) = \frac{3}{2\sqrt{3x+1}}$$

7 Area Between Curves

Find the area bounded between y = cosx and y = 2 - cosx, with $0 \leq x \leq 2\pi$

See figure 3 for the graph.

The general formula for the area between two curves is

$$Area = \int_a^b [f(x) - g(x)]\, dx$$

where a and b are the x values where the area begins and ends, f(x) is the top function, and g(x) is the bottom function.

The first thing to find is the a and b values. When looking at the area between two curves, these are the points where the two functions *intersect*. To find the intersection points we set the functions equal to each other:

$$cosx = 2 - cosx$$

$$2cosx = 2$$

$$cosx = 1$$

$$x = 0, 2\pi$$

Thus, $a = 0$ and $b = 2\pi$.

Looking at figure 3, we can see that the top curve is $y = 2 - cosx$ and the bottom is $y = cosx$, so the area formula is:

$$Area = \int_0^{2\pi} [(2 - cosx) - cosx]\, dx$$

$$= \int_0^{2\pi} [2 - 2cosx]\, dx$$

$$= (2x - 2sinx)|_0^{2\pi}$$

$$= [2(2\pi) - 2sin(2\pi)] - [2(0) - 2sin(0)]$$

$$= 4\pi$$

8 Curve Sketching

For the function $f(x) = \frac{x}{x^2+4}$, find all intercepts, asymptotes, local extrema and inflection points. Use this information to sketch a detailed graph of $y = f(x)$.

x intercepts happen when y = f(x) = 0:

$$0 = \frac{x}{x^2 + 4}$$

$$0 = x$$

So there is an x intercept at the point (0,0). (This is also a y intercept!)

y intercepts are found by setting x = 0:

$$f(0) = \frac{0}{0^2 + 4} = 0$$

So again we find the point (0,0). Thus the only intercept is the point at the origin.

Vertical asymptotes around found at values of x which are excluded from the domain. Since we are dealing with a rational function, any values of x not excluded from the domain will be values of x which make the denominator, $x^2 + 4$, equal to zero:

$$x^2 + 4 = 0$$
$$x^2 = -4$$

There are no *real* values of x which satisfy this, so there are no points excluded from the domain. As such, we don't have any vertical asymptotes.

We find horizontal asymptotes by looking at the behaviour of x far away from the origin. Mathematically, we are looking at

$$\lim_{x \to \infty} \frac{x}{x^2 + 4}$$

and

$$\lim_{x \to -\infty} \frac{x}{x^2 + 4}$$

Like before, we factor out the highest power of x in the top, as well as the highest power in the bottom. Let's look at the limit to positive ∞:

$$\lim_{x \to \infty} \frac{x}{x^2 + 4} = \lim_{x \to \infty} \frac{x}{(x^2)\left(1 + \frac{4}{x^2}\right)} = \lim_{x \to \infty} \frac{1}{(x)\left(1 + \frac{4}{x^2}\right)}$$
$$= \left(\lim_{x \to \infty} \frac{1}{x}\right)\left(\lim_{x \to \infty} \frac{1}{1 + \frac{4}{x^2}}\right)$$
$$= 0 \cdot 1$$
$$= 0$$

And the limit with $-\infty$ is identical, giving us zero as well.

Horizontal asymptotes are in the form of y = c, so there is a horizontal asymptote at y = 0 (the x axis). Note: if this limit gives you $\pm\infty$, it means there are no horizontal asymptotes.

To find the local extrema, we have to find the first derivative of the function.

$$f'(x) = \frac{1 \cdot (x^2 + 4) - x(2x)}{(x^2 + 4)^2}$$

$$f'(x) = \frac{4 - x^2}{(x^2 + 4)^2}$$

While we're at it, let's find the second derivative (it will be useful in a moment):

$$f''(x) = \frac{-2x(x^2 + 4)^2 - 2(x^2 + 4)(2x)(4 - x^2)}{(x^2 + 4)^4}$$

$$f''(x) = \frac{-2x(x^2+4) - 2(2x)(4-x^2)}{(x^2+4)^3}$$

$$f''(x) = \frac{-2x^3 - 8x - 16x + 4x^3}{(x^2+4)^3}$$

$$f''(x) = \frac{2x^3 - 24x}{(x^2+4)^3}$$

$$f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}$$

Critical points are found where $f'(x) = 0$, so we'll set the first derivative to zero and solve for x:

$$0 = \frac{4-x^2}{(x^2+4)^2}$$

$$0 = 4 - x^2$$

$$x = \pm 2$$

To see if these x values are minima or maxima, we use the second derivative test (minima for $f''(x) > 0$, maxima for $f''(x) < 0$):

$$f''(2) = \frac{2(2)[(2)^2 - 12]}{(2^2+4)^3}$$

$$f''(2) = \frac{-32}{8^3} < 0$$

So there is a maxima at x = 2. Plugging this x value into the original function, we find the point of the maxima to be $(2, \frac{1}{4})$.

$$f''(-2) = \frac{2(-2)[(-2)^2 - 12]}{((-2)^2+4)^3}$$

$$f''(-2) = \frac{32}{8^3} > 0$$

So there is a minima at x = -2. The corresponding point is $(-2, \frac{-1}{4})$.

In the same style as finding relative maxima, inflection points are found by looking where f"(x) = 0.

$$0 = \frac{2x(x^2-12)}{(x^2+4)^3}$$

$$0 = 2x(x^2-12)$$

$$x = 0, x = \pm\sqrt{12}$$

It turns out the sign of f"(x) changes at all three of these points, so they are all inflection points. (See figure 4). The corresponding points are (by plugging these x values into the original function):

$$(-\sqrt{12}, \frac{-\sqrt{3}}{8}), (0, 0), (\sqrt{12}, \frac{\sqrt{3}}{8})$$

With all of this information, we are ready to graph! A good rule of thumb is to plot all the points you have first (intercepts, relative extrema, inflection points) and then connect the dots, while paying attention to your asymptotes, concavity and relative extrema.

Finally, see figure 5 for a graph of this function.

Good luck with your exam!